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Lie symmetries for two-dimensional charged-particle motion

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Abstract. We present a Lie point symmetry analysis for non-relativistic two-dimensional charged-particle motion. The resulting symmetries comprise a quasi-invariance transformation, a time-dependent rotation, a time-dependent spatial translation and a dilation. We also find that the associated electromagnetic fields must satisfy a system of first-order linear partial differential equations. This system is solved exactly, yielding four classes of compatible electromagnetic fields.

1. Introduction

Lorentz equations for non-relativistic charged-particle motion constitute a very basic dynamical system whose symmetry structure is worth a detailed investigation. In a previous paper [1], we studied the Noether point symmetries for two-dimensional non-relativistic charged-particle motion. Here we make a systematic search for the Lie symmetries associated with the two-dimensional non-relativistic motion of a test particle in electromagnetic fields generated by appropriate charge and current densities. The reason for this study is two-fold. First, in any given problem, the Lie point symmetry group is more general [2] and contains the Noether point symmetry group. Second, even though Lie symmetry analysis does not yield first integrals as directly as Noether's theorem, it does open up the possibility of a reduction in the number of variables in the system. Reduction of variables is important for saving computation time in numerical simulations. For example, in the numerical analysis of the Vlasov–Maxwell system of collisionless plasma physics [3, 4], a highly desirable feature is a knowledge of the most general electromagnetic field configuration having Lie symmetry. In our study we consider the planar non-relativistic motion of a test particle under an initially general electromagnetic field. The strategy is to choose the field and source configurations later in compliance with the expected symmetry of the motion. For electromagnetic fields compatible with planar motion, the Lorentz equations are

$$\ddot{x} = E_1(x, y, t) + \dot{y}B(x, y, t) \qquad (1)$$

$$\ddot{y} = E_2(x, y, t) - \dot{x}B(x, y, t) \qquad (2)$$

where $\boldsymbol{E} = (E_1(x, y, t), E_2(x, y, t), 0)$ is the planar electric field and $\boldsymbol{B} = (0, 0, B(x, y, t))$ is the perpendicular magnetic field. Unlike in the usual approach of direct Lie symmetry analysis of charged-particle motion [5], we do not start with a prescribed electromagnetic field. Rather, we follow an inverse route and search for the conditions on $\boldsymbol{E}$ and $\boldsymbol{B}$ that imply the system (1) and (2) does admit a Lie symmetry. With this strategy in mind, we do not focus

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on any particular electromagnetic field but rather search for the most general form that may produce the symmetry. Once the general forms are known, they can eventually be specified in more detail to fit some particular application. In fact, electromagnetic fields written in terms of arbitrary functions are of fundamental importance in treating the Vlasov–Maxwell system in collisionless plasma physics [3,4]. Certain categories of fields are too simple to serve the needs of such specific applications. We therefore do not consider, in the following, those choices implying a too restricted field configuration, such as those being homogeneous in space. This strategy excludes some of the possible symmetries. Nevertheless, the approach gives very general electromagnetic fields, containing a substantial number of arbitrary functions. These functions can be fixed later to fit particular applications, such as those in plasma physics.

In the present case, and for the proposed working strategy, the Lie symmetry approach produces a system of linear, first-order partial differential equations for the electromagnetic fields. This system is solved by a procedure similar to that used in [1]. That is, we find the canonical group coordinates for the various symmetries and express the resulting system of partial differential equations in these coordinates. As a consequence, the system is transformed into a set of ordinary differential equations that can be solved easily. This procedure shows that, in our case, finding classes of electromagnetic fields compatible with Lie symmetry is equivalent to finding canonical group variables.

This paper is organized as follows. In section 2, we obtain the most general form of the Lie symmetries associated with planar, non-relativistic charged-particle motion. In the same section, we obtain the system of equations satisfied by the corresponding electromagnetic fields. Section 3 is dedicated to the calculation of the canonical group variables for the admissible Lie symmetries: we find four classes of canonical coordinates. In section 4, the basic system of partial differential equations satisfied by the electromagnetic fields is solved for each of the four classes of canonical group variables. In section 5, we discuss the symmetry reduction and the existence of constants of motion for the planar Lorentz equations possessing Lie symmetries. Section 6 is devoted to the conclusions.

2. Lie point symmetries

Let us consider infinitesimal point transformations:

$$\bar{x} = x + \varepsilon\eta_1(x, y, t) \tag{3}$$

$$\bar{y} = y + \varepsilon\eta_2(x, y, t) \tag{4}$$

$$\bar{t} = t + \varepsilon\tau(x, y, t) \tag{5}$$

where ε is an infinitesimal parameter. As indicated, we consider only *point* transformations and consequently, throughout the rest of this paper, Lie point symmetries and Noether point symmetries are simply referred to as Lie symmetries and Noether symmetries, respectively. Also for future convenience, we denote the generator of the group of symmetries associated with (4)–(6) by

$$G = \tau\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial x} + \eta_2\frac{\partial}{\partial y}. \tag{6}$$

The generator G appears frequently in what follows and is useful in the definition of canonical group coordinates, which play a central role in the systematic determination of the electromagnetic fields associated with the symmetries. In terms of G, the condition for Lie symmetry is

$$G^{[2]}(\boldsymbol{N})_{\boldsymbol{N}=0} = 0 \tag{7}$$

where $\boldsymbol{N} = (N_1, N_2)$,

$$N_1 = \ddot{x} - E_1(x, y, t) - \dot{y}B(x, y, t) \tag{8}$$

$$N_2 = \ddot{y} - E_2(x, y, t) + \dot{x}B(x, y, t) \tag{9}$$

$G^{[2]}$ being the generator of the twice extended group (see, for instance, [6,7]). The calculation of Lie symmetries is a fairly direct procedure and we only outline its main steps here. By inserting the equations of motion into the Lie symmetry criterion (7) we obtain a polynomial equation in the velocity components. Condition (7), being a polynomial form, implies that the coefficients of all monomials of the form $\dot{x}^m\dot{y}^n$ must be identically zero. This yields a system of partial differential equations satisfied by τ, η_1 and η_2. For instance, cubic terms give a system with a general solution

$$\tau = \rho^2(t) + g_1(t)x + g_2(t)y \tag{10}$$

where ρ, g_1 and g_2 are arbitrary functions of time. Equation (10) will be taken into account in the following.

The terms quadratic in the velocity now yield

$$\eta_{1xx} - 2\dot{g}_1 + g_2B = 0 \qquad \eta_{1xy} - \dot{g}_2 - g_1B = 0 \tag{11}$$

$$\eta_{1yy} - g_2B = 0 \qquad \eta_{2xx} + g_1B = 0 \tag{12}$$

$$\eta_{2xy} - \dot{g}_1 + g_2B = 0 \qquad \eta_{2yy} - 2\dot{g}_2 - g_1B = 0 \tag{13}$$

where we have used subscripts to denote partial derivatives. Direct inspection shows that the choice

$$g_1 = g_2 = 0 \tag{14}$$

leaves B arbitrary and implies that η_1 and η_2 are linear functions of position:

$$\eta_1 = g_3(t)x + g_4(t)y + a_1(t) \tag{15}$$

$$\eta_2 = g_5(t)x + g_6(t)y + a_2(t) \tag{16}$$

with g_3, g_4, g_5, g_6, a_1 and a_2 being functions of time only. It is important to stress now that (14) implies no restriction on the magnetic field, which remains arbitrary. Moreover, a detailed calculation involving equations (11)–(13) shows that (14) is necessary for keeping the spatial dependence in the magnetic field. In fact, we are interested in classes of magnetic fields more general than simply those homogeneous in space. Hence, we adopt (14) and the corresponding solutions (15) and (16) for η_1 and η_2. An important point here is that, up to this stage, we preserved the complete arbitrariness of the magnetic field.

To proceed we notice that the terms linear in velocity yield

$$(g_4 + g_5)B = -2(\rho\ddot{\rho} + \dot{\rho}^2) + 2\dot{g}_3 \tag{17}$$

$$(g_4 + g_5)B = 2(\rho\ddot{\rho} + \dot{\rho}^2) - 2\dot{g}_6 \tag{18}$$

$$GB = (g_3 - g_6 - 2\rho\dot{\rho})B + 2\dot{g}_4 \tag{19}$$

$$GB = (g_6 - g_3 - 2\rho\dot{\rho})B - 2\dot{g}_5 \tag{20}$$

where G is the generator defined in (6).

An examination of equations (17) and (18) shows that to keep the space dependence in the magnetic field we must choose

$$g_4 = -g_5 = -\Omega(t) \tag{21}$$

where Ω is a function of time only. Moreover, this implies, from (17) and (18), that

$$g_3 = \rho\dot{\rho} + k_1 \qquad g_6 = \rho\dot{\rho} + k_2 \tag{22}$$

with k_1 and k_2 numerical constants. Equation (22) and compatibility between equations (19) and (20) give

$$(k_1 - k_2)B = 0. \tag{23}$$

For B not identically zero, the conclusion is

$$k_1 = k_2 = k \tag{24}$$

where k is a numerical constant. Equations (19) and (20) now yield

$$GB = -2\rho\dot{\rho}B - 2\dot{\Omega} \tag{25}$$

where the generator of Lie symmetries splits into four components:

$$G = G_Q + G_R + G_T + G_S. \tag{26}$$

In equation (26),

$$G_Q = \rho^2(t)\frac{\partial}{\partial t} + \rho\dot{\rho}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \tag{27}$$

is the generator of quasi-invariance transformations [8],

$$G_R = \Omega(t)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{28}$$

generates time-dependent rotations,

$$G_T = a_1(t)\frac{\partial}{\partial x} + a_2(t)\frac{\partial}{\partial y} \tag{29}$$

generates time-dependent spatial translations, and

$$G_S = k\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \tag{30}$$

is the generator of dilations. Comparison with the generator of Noether symmetries for two-dimensional non-relativistic charged-particle motion [1] shows that the Lie symmetry generator has an additional term, depending on the parameter k. We also observe that this form is essentially new and cannot be expressed in terms of the generators G_Q, G_R and G_T.

Finally the velocity-independent terms in the Lie invariance condition yield the equations for the electric field:

$$\begin{aligned} GE_1 = (-3\rho\dot{\rho} + k)E_1 - \Omega E_2 - ((\rho\ddot{\rho} + \dot{\rho}^2)y + \dot{\Omega}x + \dot{a}_2)B \\ +(\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho})x - \ddot{\Omega}y + \ddot{a}_1 \end{aligned} \tag{31}$$

$$\begin{aligned} GE_2 = (-3\rho\dot{\rho} + k)E_2 + \Omega E_1 + ((\rho\ddot{\rho} + \dot{\rho}^2)x - \dot{\Omega}y + \dot{a}_1)B \\ +(\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho})y + \ddot{\Omega}x + \ddot{a}_2. \end{aligned} \tag{32}$$

Let us summarize the results obtained so far. Essentially, in our treatment, we excluded the excessively restricted class of spatially homogeneous magnetic fields depending only on time. This approach yields the system of equations (25), (31) and (32) satisfied by the electromagnetic fields associated with the Lie symmetries of the planar charged-particle motion. The symmetry generator in equations (25), (31) and (32), which constitute a system of linear, first-order, partial differential equations for E_1, E_2 and B, is given by (26). In comparison with the treatment of two-dimensional non-relativistic charged-particle motion with Noether symmetries, we find that Lie symmetries have an extra component, corresponding to scale transformation. This extra contribution modifies both the generator of symmetries and the equations that determine the electromagnetic fields. Notice that the basic system (25), (31) and (32) involves the electromagnetic fields and not the potentials. Thus, as in the case of Noether's symmetries, gauge invariance plays no particular role in the procedure.

In the remainder of this paper, we shall be essentially concerned with the solutions of (25), (31) and (32). These solutions will give the most general electromagnetic field under which the planar motion of test particles possesses Lie symmetry. It is useful to remark that the equation satisfied by B is decoupled from E_1 and E_2, whereas the equations for the electric field depend on B. Thus, we must first solve (25) for B and only afterwards tackle (31) and (32) for the electric field. Finally, if these solutions should constitute true electromagnetic fields, they must satisfy the homogeneous Maxwell equations. The non-homogeneous Maxwell equations can always be satisfied for a suitable choice of charge and current densities. Gauss's law is trivially verified by the magnetic field $(0, 0, B(x, y, t))$. Therefore the only requirement left is Faraday's law, which implies the extra constraint

$$E_{2x} - E_{1y} + B_t = 0. \tag{33}$$

The analysis of the system (25), (31) and (32), and the search for its solutions, are much more easily performed in canonical group coordinates. These variables are determined in the section that follows.

3. Canonical group coordinates

Canonical group coordinates [6, 7] are defined by stipulating that the symmetry transformation behaves merely like time translations. This means that, in canonical group coordinates $(\bar{x}, \bar{y}, \bar{t})$,

$$G = \frac{\partial}{\partial \bar{t}} \tag{34}$$

where $\bar{t}$ is the new time parameter. Canonical group coordinates, therefore, satisfy

$$G\bar{x} = 0 \qquad G\bar{y} = 0 \qquad G\bar{t} = 1. \tag{35}$$

This set of uncoupled linear partial differential equations can be solved by the method of characteristics for the generator (26). We find four classes of solutions, listed in the following sections.

3.1. The case $\rho \neq 0$

For $\rho \neq 0$, it is convenient to write

$$a_1 = \rho^2\dot{\alpha}_1 - (\rho\dot{\rho} + k)\alpha_1 \tag{36}$$

$$a_2 = \rho^2\dot{\alpha}_2 - (\rho\dot{\rho} + k)\alpha_2 \tag{37}$$

with suitable functions $\alpha_1(t)$ and $\alpha_2(t)$, which are themselves defined in terms of a_1 and a_2. In terms of (36) and (37), we have the following canonical group coordinates:

$$\bar{t} = \int^t \mathrm{d}\mu/\rho^2(\mu) \tag{38}$$

$$\bar{x} = \frac{\mathrm{e}^{-k\bar{t}}}{\rho}((x - \alpha_1)\cos T + (y - \alpha_2)\sin T) + \delta_1 \tag{39}$$

$$\bar{y} = \frac{\mathrm{e}^{-k\bar{t}}}{\rho}(-(x - \alpha_1)\sin T + (y - \alpha_2)\cos T) + \delta_2 \tag{40}$$

where the new functions $T = T(t)$, $\delta_1 = \delta_1(t)$ and $\delta_2 = \delta_2(t)$ are defined by

$$T(t) = \int^t d\mu\, \Omega(\mu)/\rho^2(\mu) \tag{41}$$

$$\delta_1(t) = -\int^t d\mu \frac{\Omega(\mu)}{\rho^3(\mu)} e^{-k\bar{t}(\lambda)}(\alpha_1(\mu)\sin T(\mu) - \alpha_2(\mu)\cos T(\mu)) \tag{42}$$

$$\delta_2(t) = -\int^t d\mu \frac{\Omega(\mu)}{\rho^3(\mu)} e^{-k\bar{t}(\lambda)}(\alpha_1(\mu)\cos T(\mu) + \alpha_2(\mu)\sin T(\mu)). \tag{43}$$

Note that the requirement $\rho \neq 0$ is essential, for otherwise the canonical group variables (38)–(40) would not be well defined.

For $k = \Omega = \alpha_1 = \alpha_2 = 0$, equations (38)–(40) become a quasi-invariance transformation [8]. In the general case, however, the transformations also comprise dilation, a time-dependent rotation and a time-dependent translation. To compare with the Noether symmetry approach, these canonical group variables are in direct correspondence with case 3.1 of [1]. In fact, when $k = 0$, formulae (38)–(40) become formulae (42)–(44) of [1].

3.2. The case $\rho = k = 0$ and $\Omega \neq 0$

In this case, we have only a Noether symmetry in the canonical group variables

$$\bar{t} = \frac{1}{\Omega}\tan^{-1}\left(\frac{y - \beta_2}{x - \beta_1}\right) \tag{44}$$

$$\bar{x} = ((x - \beta_1)^2 + (y - \beta_2)^2)^{1/2} \tag{45}$$

$$\bar{y} = t \tag{46}$$

with

$$\beta_1 = \beta_1(t) = -a_2/\Omega \qquad \beta_2 = \beta_2(t) = a_1/\Omega. \tag{47}$$

The variables $\bar{x}$ and $\bar{t}$ are translated polar coordinates with time playing the role of an azimuthal angle and $\bar{x}$ the role of a radial coordinate.

3.3. The case $\rho = k = \Omega = 0$

Again, we have only Noether symmetry. Written in more symmetrical notation as compared to that in [1], the canonical group variables are

$$\bar{t} = \frac{a_1 x + a_2 y}{a_1^2 + a_2^2} \tag{48}$$

$$\bar{x} = a_2 x - a_1 y \tag{49}$$

$$\bar{y} = t. \tag{50}$$

Notice that no singularity occurs since the denominator $a_1^2 + a_2^2 \neq 0$ for non-trivial symmetries.

3.4. The case $\rho = 0$, $k \neq 0$

In this case the canonical coordinates are

$$\bar{t} = \frac{1}{2k}\log((x - \gamma_1)^2 + (y - \gamma_2)^2) \tag{51}$$

$$\bar{x} = \arctan\left(\frac{y - \gamma_2}{x - \gamma_1}\right) - \Omega\bar{t} \tag{52}$$

$$\bar{y} = t \tag{53}$$

where

$$\gamma_1 = -\frac{ka_1 + \Omega a_2}{k^2 + \Omega^2} \qquad \gamma_2 = \frac{\Omega a_1 - ka_2}{k^2 + \Omega^2}. \tag{54}$$

This symmetry transformation represents a dilation, plus a time-dependent rotation and translation.

4. Electromagnetic fields

We now consider equations (25), (31) and (32) for the electromagnetic fields for each of the four possible symmetry transformations, expressed in canonical group variables.

4.1. The case $\rho \neq 0$

Equation (25) becomes, in the canonical group coordinates (38)–(40):

$$B_{\bar{t}} = -\frac{2\rho'}{\rho}B - \frac{2\Omega'}{\rho^2} \tag{55}$$

where a prime denotes total differentiation with respect to $\bar{t}$. The general solution for (55) is

$$B = -\frac{2\Omega}{\rho^2} + \frac{1}{\rho^2}\bar{B}(\bar{x}, \bar{y}) \tag{56}$$

where $\bar{B}(\bar{x}, \bar{y})$ is an arbitrary function of the indicated arguments. Notice that the resulting magnetic field depends on the spatial coordinates through $\bar{x}$ and $\bar{y}$ and, therefore, is in general non-homogeneous. This is a significant improvement on the earlier known result [9]. Formally, B is identical to the magnetic field of case 4.1 in [1] on Noether symmetries. The difference is in the form of canonical group variables.

To find the corresponding electric field, we must solve system (31) and (32) by taking the solution (56) into account. Here, it is useful to introduce the quantities Σ_1 and Σ_2, defined by

$$\Sigma_1 = \rho^3 e^{-k\bar{t}}(E_1 \cos T + E_2 \sin T) \tag{57}$$

$$\Sigma_2 = \rho^3 e^{-k\bar{t}}(-E_1 \sin T + E_2 \cos T) \tag{58}$$

representing a rotation and a rescaling of the electric field which, in this case, can be viewed as a circularly polarized wave with time-dependent amplitude. In these new variables, the system (31) and (32) decouples and can be cast into

$$\frac{\partial \Sigma_1}{\partial \bar{t}} = \frac{\partial \psi_1}{\partial \bar{t}} \qquad \frac{\partial \Sigma_2}{\partial \bar{t}} = \frac{\partial \psi_2}{\partial \bar{t}} \tag{59}$$

where

$$\begin{aligned}\psi_1 = {} & \left(-\frac{\rho'}{\rho}(\bar{y} - \delta_2) + \delta_2' + k\delta_2 - \Omega(\bar{x} - \delta_1) + \frac{e^{-k\bar{t}}}{\rho}(\alpha_1' \sin T - \alpha_2' \cos T)\right)\bar{B}(\bar{x}, \bar{y}) \\ & + \left(\frac{\rho''}{\rho} - 2\frac{\rho'^2}{\rho^2} + \Omega^2\right)(\bar{x} - \delta_1) - \left(\Omega' - 2\frac{\rho'}{\rho}\Omega\right)(\bar{y} - \delta_2) \\ & + \frac{e^{-k\bar{t}}}{\rho}\left(\Omega'\alpha_1 - \Omega\left(\alpha_1' + \frac{\rho'}{\rho}\alpha_1\right) + \alpha_2'' - 2\frac{\rho'}{\rho}\alpha_2' + \Omega^2\alpha_2\right)\sin T \\ & + \frac{e^{-k\bar{t}}}{\rho}\left(-\Omega'\alpha_2 + \Omega\left(\alpha_2' + \frac{\rho'}{\rho}\alpha_2\right) + \alpha_1'' - 2\frac{\rho'}{\rho}\alpha_1' + \Omega^2\alpha_1\right)\cos T - k(\delta_1' + k\delta_1)\end{aligned} \tag{60}$$

$$\psi_2 = \left(+\frac{\rho'}{\rho}(\bar{x}-\delta_1) - \delta_1' - k\delta_1 - \Omega(\bar{y}-\delta_2) + \frac{e^{-k\bar{t}}}{\rho}(\alpha_1'\cos T + \alpha_2'\sin T)\right)\bar{B}(\bar{x},\bar{y})$$
$$+\left(\frac{\rho''}{\rho} - 2\frac{\rho'^2}{\rho^2} + \Omega^2\right)(\bar{y}-\delta_2) + \left(\Omega' - 2\frac{\rho'}{\rho}\Omega\right)(\bar{x}-\delta_1)$$
$$-\frac{e^{-k\bar{t}}}{\rho}\left(-\Omega'\alpha_2 + \Omega\left(\alpha_2' + \frac{\rho'}{\rho}\alpha_2\right) + \alpha_1'' - 2\frac{\rho'}{\rho}\alpha_1' + \Omega^2\alpha_1\right)\sin T$$
$$+\frac{e^{-k\bar{t}}}{\rho}\left(+\Omega'\alpha_1 - \Omega\left(\alpha_1' + \frac{\rho'}{\rho}\alpha_1\right) + \alpha_2'' - 2\frac{\rho'}{\rho}\alpha_2' + \Omega^2\alpha_2\right)\cos T - k(\delta_2' + k\delta_2). \tag{61}$$

The general solution for (59) is

$$\Sigma_1 = \psi_1 + \bar{E}_1(\bar{x},\bar{y}) \qquad \Sigma_2 = \psi_2 + \bar{E}_2(\bar{x},\bar{y}) \tag{62}$$

where, as indicated, $\bar{E}_1$ and $\bar{E}_2$ do not depend on $\bar{t}$.

To obtain the electric field in the original variables we use the inverse of the transformation (57) and (58):

$$E_1 = \frac{e^{k\bar{t}}}{\rho^3}(\Sigma_1\cos T - \Sigma_2\sin T) \tag{63}$$
$$E_2 = \frac{e^{k\bar{t}}}{\rho^3}(\Sigma_1\sin T + \Sigma_2\cos T). \tag{64}$$

Substitution of equation (62) into (63) and (64), and back-transformation to the original variables (x, y, t), yields the electric field

$$E_1 = \ddot{\alpha}_1 + \frac{\ddot{\rho}}{\rho}(x-\alpha_1) + \frac{\Omega^2 x}{\rho^4} - (\rho\dot{\Omega} - 2\dot{\rho}\Omega)\frac{y}{\rho^3} + \frac{\Omega}{\rho^3}(\rho\dot{\alpha}_2 - \dot{\rho}\alpha_2)$$
$$+\frac{k^2 e^{k\bar{t}}}{\rho^3}(\delta_2\sin T - \delta_1\cos T) - \frac{k\Omega\alpha_2}{\rho^4}$$
$$+\frac{e^{k\bar{t}}}{\rho^3}(\bar{E}_1(\bar{x},\bar{y})\cos T - \bar{E}_2(\bar{x},\bar{y})\sin T)$$
$$-\frac{1}{\rho^4}(\rho\dot{\rho}(y-\alpha_2) + \rho^2\dot{\alpha}_2 + \Omega x - k\rho e^{k\bar{t}}(\delta_2\cos T + \delta_1\sin T))\bar{B}(\bar{x},\bar{y}) \tag{65}$$

$$E_2 = \ddot{\alpha}_2 + \frac{\ddot{\rho}}{\rho}(y-\alpha_2) + \frac{\Omega^2 y}{\rho^4} + (\rho\dot{\Omega} - 2\dot{\rho}\Omega)\frac{x}{\rho^3} - \frac{\Omega}{\rho^3}(\rho\dot{\alpha}_1 - \dot{\rho}\alpha_1)$$
$$-\frac{k^2 e^{k\bar{t}}}{\rho^3}(\delta_2\cos T + \delta_1\sin T) + \frac{k\Omega\alpha_1}{\rho^4}$$
$$+\frac{e^{k\bar{t}}}{\rho^3}(\bar{E}_2(\bar{x},\bar{y})\cos T + \bar{E}_1(\bar{x},\bar{y})\sin T)$$
$$+\frac{1}{\rho^4}(\rho\dot{\rho}(x-\alpha_1) + \rho^2\dot{\alpha}_1 - \Omega y - k\rho e^{k\bar{t}}(\delta_1\cos T - \delta_2\sin T))\bar{B}(\bar{x},\bar{y}). \tag{66}$$

We still need to consider Faraday's law, which, in our case, is equivalent to equation (33). A detailed calculation using the magnetic field (56) and the electric field (65) and (66), reduces Faraday's law to

$$\bar{E}_{2\bar{x}} - \bar{E}_{1\bar{y}} = k(\bar{x}\bar{B}_{\bar{x}} + \bar{y}\bar{B}_{\bar{y}}). \tag{67}$$

For $k = 0$ (the Noether symmetry subcase), equation (67) becomes $\bar{E}_{2\bar{x}} - \bar{E}_{1\bar{y}} = 0$ with the general solution

$$\bar{E}_1 = -\frac{\partial}{\partial \bar{x}} \bar{V}(\bar{x}, \bar{y}) \qquad \bar{E}_2 = -\frac{\partial}{\partial \bar{y}} \bar{V}(\bar{x}, \bar{y}) \tag{68}$$

where $\bar{V}(\bar{x}, \bar{y})$ is an arbitrary function of the indicated argument. For $k = 0$, $\bar{B}$ remains arbitrary. For $k \neq 0$, equation (67) is a supplementary constraint imposed on the electromagnetic field. Indeed, for $k \neq 0$, (67) has the general solution

$$\bar{E}_1 = \sigma_1 - \frac{\bar{y}}{2}(\sigma_{1\bar{y}} - \sigma_{2\bar{x}}) \tag{69}$$

$$\bar{E}_2 = \sigma_2 + \frac{\bar{x}}{2}(\sigma_{1\bar{y}} - \sigma_{2\bar{x}}) \tag{70}$$

$$\bar{B} = \frac{1}{2k}(\sigma_{1\bar{y}} - \sigma_{2\bar{x}}) \tag{71}$$

where $\sigma_1 = \sigma_1(\bar{x}, \bar{y})$ and $\sigma_2 = \sigma_2(\bar{x}, \bar{y})$ are arbitrary functions.

In conclusion, we have obtained a very general class of electromagnetic fields yielding Lie symmetries in the planar Lorenz equation. The magnetic field is given by equation (56) and the electric field by (65), (66) and condition (67). The electromagnetic field involves several arbitrary functions, namely $\rho(t), \alpha_1(t), \alpha_2(t), \Omega(t), \bar{B}(\bar{x}, \bar{y}), \bar{E}_1(\bar{x}, \bar{y})$ and $\bar{E}_2(\bar{x}, \bar{y})$, subjected to constraint (67). The coordinates $\bar{x}$ and $\bar{y}$ are defined by (39) and (40), respectively.

4.2. *The case $\rho = 0$, $k = 0$ and $\Omega \neq 0$*

In this case we also have only Noether symmetry. Hence, we simply quote the main results from [1]. The electromagnetic field is given by

$$B = \bar{B}(\bar{x}, \bar{y}) \tag{72}$$

$$E_1 = \ddot{\beta}_1 - \dot{\beta}_2 \bar{B}(\bar{x}, \bar{y}) + (x - \beta_1)\bar{E}_1(\bar{x}, \bar{y}) - (y - \beta_2)\bar{E}_2(\bar{x}, \bar{y}) \tag{73}$$

$$E_2 = \ddot{\beta}_2 + \dot{\beta}_1 \bar{B}(\bar{x}, \bar{y}) + (x - \beta_1)\bar{E}_2(\bar{x}, \bar{y}) + (y - \beta_2)\bar{E}_1(\bar{x}, \bar{y}) \tag{74}$$

where $\bar{B}$, $\bar{E}_1$ and $\bar{E}_2$ are arbitrary functions of $\bar{x}$ and $\bar{y}$ given in (45) and (46). Faraday's law requires additionally that

$$\bar{x}\bar{E}_{2\bar{x}} + 2\bar{E}_2 = -\bar{B}_{\bar{y}} \tag{75}$$

whose solution is

$$\bar{E}_2 = -\frac{1}{\bar{x}^2}\frac{\partial \psi}{\partial \bar{y}} \qquad \bar{B} = \frac{1}{\bar{x}}\frac{\partial \psi}{\partial \bar{x}} \tag{76}$$

for arbitrary $\psi = \psi(\bar{x}, \bar{y})$.

In conclusion, the electromagnetic field is given by (72)–(74) under the additional constraint (76). There remain four arbitrary functions, namely $E_1(\bar{x}, \bar{y})$, $\psi(\bar{x}, \bar{y})$, $\beta_1(t)$ and $\beta_2(t)$, with $\bar{x}$, $\bar{y}$ defined in (45) and (46). We also note that, in the present case, $\Omega(t)$ has to be constant in order to produce a physically meaningful electromagnetic field (for details, see [1]). Without loss of generality, we take $\Omega = 1$.

4.3. *The case $\rho = k = \Omega = 0$*

From [1], written in a more symmetric notation, the electromagnetic fields are

$$B = \bar{B}(\bar{x}, \bar{y}) \tag{77}$$

$$E_1 = \left(\frac{a_1x + a_2y}{a_1^2 + a_2^2}\right)(\ddot{a}_1 - \dot{a}_2\bar{B}(\bar{x}, \bar{y})) + \bar{E}_1(\bar{x}, \bar{y}) \tag{78}$$

$$E_2 = \left(\frac{a_1x + a_2y}{a_1^2 + a_2^2}\right)(\ddot{a}_2 + \dot{a}_1\bar{B}(\bar{x}, \bar{y})) + \bar{E}_2(\bar{x}, \bar{y}) \tag{79}$$

where $\bar{B}$, $\bar{E}_1$ and $\bar{E}_2$ are arbitrary functions and $\bar{x}$ and $\bar{y}$ are given in (49), (50).

The solution of the equation resulting from the symmetry condition must again verify the constraint imposed by Faraday's law. In this case, the condition implies

$$\bar{B} = \psi_{\bar{x}} \tag{80}$$

$$\bar{E}_1 = -\frac{1}{a_1^2 + a_2^2}((a_2x - a_1y)(\dot{a}_1\psi_{\bar{x}} + \ddot{a}_2) + a_1\psi_{\bar{y}}) - \bar{V}_x \tag{81}$$

$$\bar{E}_2 = -\frac{1}{a_1^2 + a_2^2}((a_2x - a_1y)(\dot{a}_2\psi_{\bar{x}} - \ddot{a}_1) + a_2\psi_{\bar{y}}) - \bar{V}_y. \tag{82}$$

Here, $\psi = \psi(\bar{x}, \bar{y})$ and $\bar{V} = \bar{V}(\bar{x}, \bar{y})$ are arbitrary functions.

This determines the class of solutions for the electromagnetic field. B is given by (77) and E_1 and E_2 are given by (78) and (79). The functions $\bar{B}$, $\bar{E}_1$ and $\bar{E}_2$ are given by (80)–(82), in terms of the arbitrary functions $\psi(\bar{x}, \bar{y})$ and $\bar{V}(\bar{x}, \bar{y})$, with $\bar{x}$ and $\bar{y}$ given by (49) and (50). The arbitrary functions $a_1(t)$ and $a_2(t)$ also enter into the definition of the electromagnetic field, and so four arbitrary functions participate in the final solution.

4.4. The case $\rho = 0$ and $k \neq 0$

In this case the equation for the magnetic field becomes

$$B_{\bar{t}} = -2\dot{\Omega}(\bar{y}) \tag{83}$$

and have the solution

$$B = -2\dot{\Omega}(\bar{y})\bar{t} + \bar{B}(\bar{x}, \bar{y}). \tag{84}$$

Inserting this magnetic field in the equations for the electric field yields

$$E_{1\bar{t}} = kE_1 - \Omega E_2 + (\dot{\Omega}x + \dot{a}_2)(2\dot{\Omega}\bar{t} - \bar{B}) - \ddot{\Omega}y + \ddot{a}_1 \tag{85}$$

$$E_{2\bar{t}} = kE_2 + \Omega E_1 + (\dot{\Omega}y - \dot{a}_1)(2\dot{\Omega}\bar{t} - \bar{B}) + \ddot{\Omega}x + \ddot{a}_2. \tag{86}$$

This system may be handled more conveniently in the new variables

$$\Sigma_1 = \mathrm{e}^{-k\bar{t}}(E_1 \cos\Omega\bar{t} + E_2 \sin\Omega\bar{t}) \tag{87}$$

$$\Sigma_2 = \mathrm{e}^{-k\bar{t}}(-E_1 \sin\Omega\bar{t} + E_2 \cos\Omega\bar{t}). \tag{88}$$

In these variables, we have the transformed equations

$$\frac{\partial\Sigma_1}{\partial\bar{t}} = \frac{\partial\psi_1}{\partial\bar{t}} \qquad \frac{\partial\Sigma_2}{\partial\bar{t}} = \frac{\partial\psi_2}{\partial\bar{t}} \tag{89}$$

where

$$\psi_1 = (\dot{\Omega}^2\bar{t}^2 - \dot{\Omega}\bar{B}\bar{t})\cos\bar{x} - \ddot{\Omega}\bar{t}\sin\bar{x} + (2\dot{\Omega}\dot{\gamma}_2\bar{t} + \ddot{\gamma}_1 - \dot{\gamma}_2\bar{B})\mathrm{e}^{-k\bar{t}}\cos\Omega\bar{t} + (-2\dot{\Omega}\dot{\gamma}_1\bar{t} + \ddot{\gamma}_2 + \dot{\gamma}_1\bar{B})\mathrm{e}^{-k\bar{t}}\sin\Omega\bar{t} \tag{90}$$

$$\psi_2 = (\dot{\Omega}^2\bar{t}^2 - \dot{\Omega}\bar{B}\bar{t})\sin\bar{x} + \ddot{\Omega}\bar{t}\cos\bar{x} + (-2\dot{\Omega}\dot{\gamma}_1\bar{t} + \ddot{\gamma}_2 + \dot{\gamma}_1\bar{B})\mathrm{e}^{-k\bar{t}}\cos\Omega\bar{t} - (2\dot{\Omega}\dot{\gamma}_2\bar{t} + \ddot{\gamma}_1 - \dot{\gamma}_2\bar{B})\mathrm{e}^{-k\bar{t}}\sin\Omega\bar{t}. \tag{91}$$

The solutions to (89) are

$$\Sigma_1 = \psi_1 + \bar{E}_1(\bar{x}, \bar{y}) \qquad \Sigma_2 = \psi_2 + \bar{E}_2(\bar{x}, \bar{y}) \tag{92}$$

and the inverse transformation for (87) and (88) is

$$E_1 = e^{k\bar{t}}(\Sigma_1 \cos\Omega\bar{t} - \Sigma_2 \sin\Omega\bar{t}) \tag{93}$$

$$E_2 = e^{k\bar{t}}(\Sigma_1 \sin\Omega\bar{t} + \Sigma_2 \cos\Omega\bar{t}). \tag{94}$$

Back in the original coordinates, the resulting electric field becomes

$$E_1 = \ddot{\gamma}_1 + 2\dot{\Omega}\dot{\gamma}_2\bar{t} - \dot{\gamma}_2\bar{B} + \dot{\Omega}\bar{t}(\dot{\Omega}\bar{t} - \bar{B})(x - \gamma_1) - \ddot{\Omega}\bar{t}(y - \gamma_2) + e^{k\bar{t}}(\bar{E}_1 \cos\Omega\bar{t} - \bar{E}_2 \sin\Omega\bar{t}) \tag{95}$$

$$E_2 = \ddot{\gamma}_2 - 2\dot{\Omega}\dot{\gamma}_1\bar{t} + \dot{\gamma}_1\bar{B} + \dot{\Omega}\bar{t}(\dot{\Omega}\bar{t} - \bar{B})(y - \gamma_2) + \ddot{\Omega}\bar{t}(x - \gamma_1) + e^{k\bar{t}}(\bar{E}_1 \sin\Omega\bar{t} + \bar{E}_2 \cos\Omega\bar{t}). \tag{96}$$

In order to obtain simpler expressions, we expressed the results in a more convenient hybrid notation involving (x, y) and the transformed time $\bar{t}$. Notice the generality kept in the electromagnetic field, which involves six arbitrary functions, namely γ_1, γ_2, Ω, $\bar{B}$, $\bar{E}_1$ and $\bar{E}_2$.

To finalize, Faraday's law imposes, in this case

$$k\frac{\partial\bar{B}}{\partial\bar{y}} = -\ddot{\Omega}(\bar{y}) + (k\sin\bar{x} - \Omega(\bar{y})\cos\bar{x})\bar{E}_1 - (k\cos\bar{x} + \Omega(\bar{y})\sin\bar{x})\bar{E}_2 + (k\cos\bar{x} - \Omega(\bar{y})\sin\bar{x})\frac{\partial\bar{E}_1}{\partial\bar{x}} + (k\sin\bar{x} + \Omega(\bar{y})\cos\bar{x})\frac{\partial\bar{E}_2}{\partial\bar{x}}. \tag{97}$$

This condition must be satisfied by the arbitrary functions appearing in the solution. For instance, after specifying Ω, $\bar{E}_1$ and $\bar{E}_2$, we can view (97) as an equation to determine $\bar{B}$, up to the addition of an arbitrary function of $\bar{x}$.

5. Symmetry reductions and invariants

In this section we study the implications of the Lie symmetries in the Lorentz equations for the planar motion of a test particle. First, and most important, in terms of canonical group variables, the equations of motion will not depend explicitly on the time parameter [6, 7]. For the electromagnetic fields discussed in section 4.1, canonical group variables are indeed convenient, as shown in the following. However, in general, for the electromagnetic fields treated in this work, the form of the equations of motion in canonical group variables is rather cumbersome, as the reader may verify. In this case, elimination of the independent variable comes at a high price and may not be worth the effort.

One very important point in the study of any dynamical system concerns the existence of invariants (constants of motion or first integrals). In the case of Noether symmetry, on the one hand, the knowledge of the symmetry generator immediately provides a conserved quantity. On the other hand, the relation between Lie symmetries and conserved quantities is more indirect [11]. In our case we tried, without success, the Lie method [6, 7] to find the constants of motion for the Lorentz equations with electromagnetic fields of the form given in section 4.4 (corresponding to a non-Noether symmetry). Thus, in this case, it seems that a more promising approach for finding invariants is the direct method [10]. We apply this technique in the next sections to discuss the existence of invariants and the integrability of the planar Lorentz equations with Lie symmetries.

5.1. The case $\rho \neq 0$

Substitute, into the Lorentz equations (1) and (2), the electromagnetic fields specified by (56), (65) and (66). Writing the equations of motion in the canonical group coordinates introduced in section 3.1 yields

$$\bar{x}'' + 2k\bar{x}' = F_1(\bar{x}, \bar{y}) + \bar{y}'\bar{B}(\bar{x}, \bar{y}) \tag{98}$$

$$\bar{y}'' + 2k\bar{y}' = F_2(\bar{x}, \bar{y}) - \bar{x}'\bar{B}(\bar{x}, \bar{y}) \tag{99}$$

where

$$F_1(\bar{x}, \bar{y}) = E_1(\bar{x}, \bar{y}) + k\bar{y}\bar{B}(\bar{x}, \bar{y}) - k^2\bar{x} \tag{100}$$

$$F_2(\bar{x}, \bar{y}) = E_2(\bar{x}, \bar{y}) - k\bar{x}\bar{B}(\bar{x}, \bar{y}) - k^2\bar{y} \tag{101}$$

are new arbitrary functions and a prime denotes differentiation with respect to $\bar{t}$.

At this point we notice that, in terms of F_1 and F_2, Faraday's constraint (67) takes on the interesting form

$$2k\bar{B} = F_{1\bar{y}} - F_{2\bar{x}}. \tag{102}$$

We also notice that (98) and (99) can be viewed as the equations for the planar non-relativistic motion of a charged particle subjected to an additional friction force proportional to the constant k. However, for non-zero $k\bar{B}$ we cannot interpret $\bar{B}$, F_1 and F_2 as the components of a static magnetic and electric field. Actually, for static electromagnetic fields, Faraday's law imposes $F_{1\bar{y}} - F_{2\bar{x}} = 0$, which is incompatible with (102) if $k\bar{B} \neq 0$. We recall that the friction term is associated with a non-Noether symmetry ($k \neq 0$), a fact that probably hinders the search for invariants. The equations of motion in canonical group coordinates are indeed autonomous but, as they stand, they are not integrable in the general case. For integrability, suitable restrictions must be imposed on the arbitrary functions F_1, F_2 and $\bar{B}$. In the following, we will study a few cases for which (98) and (99) admit an invariant. For the sake of simplicity of notation, we omit the overbars in the rest of this section.

We start with the simpler case of Noether symmetry. When $k = 0$, the energy is a first integral:

$$I = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \phi(x, y) \tag{103}$$

$\phi(x, y)$ being the scalar potential, $F_1 = -\phi_x$ and $F_2 = -\phi_y$. As expected, the energy first integral (104) is the Noether invariant derived in [1]. However, knowledge of just one invariant is not sufficient for complete integrability. In fact, according to the Liouville–Arnold theorem [12], a Hamiltonian system of two degrees of freedom needs two invariants in involution for integrability. For electromagnetic fields in general, only in rare situations is a second constant of motion known. In the literature, Hietarinta [10] and Dorizzi *et al* [13] present some cases of static electromagnetic fields for which an additional invariant, besides energy, is available.

Let us proceed with the non-Noether symmetry and consider $k \neq 0$. In this case, there is no first integral available *a priori*. In the Lie method for construction of invariants, we seek constants of motion in the form of functions that are invariant under the first extended group of symmetries (for details, see [6, 7]). In the present case, this implies searching for invariants of (98) and (99) not depending on the independent variable. In this case an useful alternative method for construction of invariants is the direct method [10], which consists of supposing an invariant of a given prescribed form. In the following, we try the direct method to search for invariants of (98) and (99). Also, to be consistent with Lie's method, we consider only time-independent invariants.

As a starting form we take the following ansatz, linear in velocities, for the invariant

$$I = f_1(x, y)\dot{x} + f_2(x, y)\dot{y} + f_3(x, y) \tag{104}$$

where f_i are functions to be determined. If I is an invariant, then by definition $\mathrm{d}I/\mathrm{d}t = 0$. Under this condition, equations (98) and (99) yield a quadratic polynomial in velocities that must be identically zero. Equating to zero the coefficients of different powers of velocities, a system of partial differential equations results, involving f_i, B and F_i, that may be easily solved. Here we find two cases for which (98) and (99) admit an invariant linear in the velocities. In the first case, B and F_i are given by

$$B = \tilde{B}(c_2x - c_1y) \tag{105}$$

$$F_1 = -\tilde{V}_x(c_2x - c_1y) + \frac{2kc_2}{c_1^2 + c_2^2}(c_1x + c_2y)\tilde{B}(c_2x - c_1y) \tag{106}$$

$$F_2 = -\tilde{V}_y(c_2x - c_1y) - \frac{2kc_1}{c_1^2 + c_2^2}(c_1x + c_2y)\tilde{B}(c_2x - c_1y) \tag{107}$$

where c_1 and c_2 are arbitrary numerical constants not simultaneously zero, and $\tilde{B}$ and $\tilde{V}$ are arbitrary functions depending on $c_2x - c_1y$. Faraday's constraint (102) is already taken into account in (105)–(107). The corresponding invariant is

$$I = c_1(\dot{x} + 2kx) + c_2(\dot{y} + 2ky) + \int^{c_2x-c_1y} \mathrm{d}\lambda\, \tilde{B}(\lambda). \tag{108}$$

For the integration of the equations of motion, define

$$\tilde{x} = c_2x - c_1y. \tag{109}$$

Using (98), (99) and (105)–(108), it is not difficult to obtain

$$\ddot{\tilde{x}} + 2k\dot{\tilde{x}} = -U_{\tilde{x}}(\tilde{x}; I) \tag{110}$$

where

$$U(\tilde{x}; I) = \tilde{V}(\tilde{x}) - \int^{\tilde{x}} \mathrm{d}\lambda\, \tilde{B}(\lambda)\left(I - \int^{\lambda} \mathrm{d}\lambda'\, \tilde{B}(\lambda')\right). \tag{111}$$

Equation (110) is in the form of an equation for the one-dimensional motion of a test particle subjected to a time-independent potential and a friction force. As is well known [11, 14], such equations can be reduced to an Abel equation of second kind, which is not generically (that is, for arbitrary U) integrable. However, for $k = 0$ (the Noether subcase), equation (110) can be reduced to a quadrature. Once this quadrature is carried out, giving $\tilde{x}$ as a function of time, we can easily reconstruct the planar motion $(x(t), y(t))$ using the invariant (108).

The second case, where (98) and (99) possess an invariant linear in velocities, is given by

$$B = 4k\theta + \tilde{B}(r) \tag{112}$$

$$F_1 = -\tilde{V}_x(r) + 2k\theta(2k\theta + \tilde{B}(r))r\cos\theta \tag{113}$$

$$F_2 = -\tilde{V}_y(r) + 2k\theta(2k\theta + \tilde{B}(r))r\sin\theta \tag{114}$$

where $\tilde{B}$ and $\tilde{V}$ are arbitrary functions of the indicated argument. Also, in (112)–(114) and in the rest of this section,

$$r = ((x - c_1)^2 + (y - c_2)^2)^{1/2} \qquad \theta = \arctan\left(\frac{y - c_2}{x - c_1}\right) \tag{115}$$

c_1 and c_2 being arbitrary numerical constants. The corresponding invariant is

$$I = r^2(\dot{\theta} + 2k\theta) + \int^{r} \mathrm{d}\lambda\, \lambda\tilde{B}(\lambda). \tag{116}$$

Notice that $\tilde{B}$ and F_i as given by (112)–(114) are multivalued functions, a fact that makes its physical interpretation rather difficult. Nevertheless, it is interesting to write the equation of motion in terms of r, yielding

$$\ddot{r} + 2k\dot{r} = -U_r(r; I). \tag{117}$$

Here

$$U(r;I)=\tilde{V}(r)-\int^{r}\frac{\mathrm{d}\lambda}{\lambda^{3}}\left(I-\int^{\lambda}\mathrm{d}\lambda'\,\lambda'\tilde{B}(\lambda')\right)^{2}-\int^{r}\mathrm{d}\lambda\frac{\tilde{B}(\lambda)}{\lambda}\left(I-\int^{\lambda}\mathrm{d}\lambda'\,\lambda'\tilde{B}(\lambda')\right). \tag{118}$$

Again, (117) describes a one-dimensional motion under a time-independent potential and a friction force, reducible to an Abel equation of second kind.

We searched without success for non-trivial invariants quadratic in velocities for (98) and (99) when $k \neq 0$. The case $k = 0$ is already treated in the literature (see [10, 13]).

In conclusion, Lorentz equations for planar non-relativistic motion with Lie symmetry and an invariant linear in velocities are not generically integrable. However, if there is a linear invariant and also Noether symmetry, the motion is integrable. This is not surprising since, in this case, two invariants are available (the linear in velocities and the Noether invariant).

5.2. The case $\rho = k = 0$ and $\Omega \neq 0$

For the electromagnetic fields described in section 4.2, Noether symmetry applies and the invariant is

$$I = r^2\dot{\theta} + \psi(r,t) \tag{119}$$

where $r = \bar{x}$, as given by (45), and

$$\theta = \arctan\left(\frac{y-\beta_2}{x-\beta_1}\right). \tag{120}$$

The functions β_1 and β_2 were defined in (47); ψ is given in (76).

Using $\bar{E}_1$ as defined in (73) and (74), we can reduce the equation of motion for the coordinate r to

$$\ddot{r} = -U_r(r,t;I) \tag{121}$$

where

$$U(r,t;I) = \frac{1}{2r^2}(I-\psi(r,t))^2 - \int^r \mathrm{d}\lambda\,\lambda\bar{E}_1(\lambda,t). \tag{122}$$

Equation (121) describes the one-dimensional motion of a particle under a time-dependent potential. Again, integrability is an exception, and a second invariant must be found. Noether symmetry at least reduces the whole dynamics to the resolution of (121), for which some results are already available. Indeed, the search for invariants for one-dimensional motion under time-dependent potentials has been the subject of intensive research in recent years [11, 15, 16]. Whenever r as a function of time can be found from (122), we may obtain θ as a function of time from the Noether invariant (119).

5.3. The case $\rho = k = \Omega = 0$

Proceeding with the notation used in section 4.3, here again we have only a Noether invariant:

$$I = a_1\dot{x} + a_2\dot{y} - \dot{a}_1 x - \dot{a}_2 y + \psi(a_2 x - a_1 y, t). \tag{123}$$

Using

$$s = \frac{a_2 x - a_1 y}{\sqrt{a_1^2 + a_2^2}} \tag{124}$$

we get

$$\ddot{s} = -U_s(s, t; I). \tag{125}$$

Here we give up writing explicitly the complicated form of $U(s, t; I)$. The important point is that, as in section 5.2, we have the equivalent of a one-dimensional motion of a particle under a time-dependent potential. A second invariant is still needed for complete integrability.

5.4. The case $\rho = 0$, $k \neq 0$

For $\rho = 0$, $k \neq 0$ we have no Noether symmetry and no associated Noether invariant. Nevertheless, we can obtain invariants for particular electromagnetic fields in the general class determined in section 4.4. For instance, we can obtain an invariant linear in the velocities. According to (84), (95) and (96), the electromagnetic fields of section 4.4 depend on the arbitrary functions $\bar{B}$, $\bar{E}_1$ and $\bar{E}_2$. With the canonical group variables defined in section 3.4, we obtain, after some simple calculations, that, for an invariant linear in the velocity to exist, necessarily

$$\bar{B} = \tilde{B}(\bar{y}) \tag{126}$$

$$\bar{E}_1 = \tilde{E}(\bar{y}) \cos \bar{x} + \frac{1}{2}\left(\dot{\tilde{B}} + \frac{\ddot{\Omega}}{k}\right) \sin \bar{x} \tag{127}$$

$$\bar{E}_2 = \tilde{E}(\bar{y}) \sin \bar{x} - \frac{1}{2}\left(\dot{\tilde{B}} + \frac{\ddot{\Omega}}{k}\right) \cos \bar{x} \tag{128}$$

where $\tilde{B}$ and $\tilde{E}$ are arbitrary functions of $\bar{y} = t$. Faraday's constraint (97) is already taken into account in (126)–(128).

The functions $\bar{B}(\bar{x}, \bar{y})$, $\bar{E}_1(\bar{x}, \bar{y})$ and $\bar{E}_1(\bar{x}, \bar{y})$, satisfying equations (126)–(128), are the only functions that produce an invariant linear in the velocity for Lorentz equations with the given symmetry. Using

$$r = ((x - \gamma_1)^2 + (y - \gamma_2)^2)^{1/2} \qquad \theta = \arctan\left(\frac{y - \gamma_2}{x - \gamma_1}\right) \tag{129}$$

where γ_1 and γ_2 are given in (97), the associated invariant becomes

$$I = r^2\dot{\theta} + \frac{1}{2}r^2\left(\tilde{B}(t) + \frac{\dot{\Omega}}{k}\right) - \frac{\dot{\Omega}}{k}r^2 \log r. \tag{130}$$

Using the invariant given by (130) we obtain

$$\ddot{r} = -U_r(r, t; I) \tag{131}$$

for the time-dependent potential,

$$U(r, t; I) = \frac{\omega^2(t)r^2}{2} + \frac{I^2}{2r^2} + \frac{I\dot{\Omega}}{k} \log r \tag{132}$$

where we have defined

$$\omega^2(t) = \frac{1}{4}\left(\tilde{B}^2(t) - \frac{\dot{\Omega}^2}{k^2} - 4\tilde{E}(t)\right). \tag{133}$$

Explicitly, the equation for r is

$$\ddot{r} + \omega^2(t)r = \frac{I^2}{r^3} - \frac{I\dot{\Omega}}{kr}. \tag{134}$$

If either $I = 0$ or $\dot{\Omega} = 0$, then (134) is Pinney's [17] equation.

For arbitrary ω, Ω and $\tilde{B}$, we have not succeeded either at obtaining a second invariant for equation (134) or at finding quadratic invariants for the Lorentz equations with the electromagnetic fields given in section 4.4. However, if

$$I\dot{\Omega} = \frac{c}{\rho^2} \tag{135}$$

where c is a numerical constant and $\rho = \rho(t)$ is a function satisfying

$$\ddot{\rho} + \omega^2(t)\rho = 0 \tag{136}$$

then there is a second invariant J given by

$$J = \frac{1}{2}(\rho\dot{r} - \dot{\rho}r)^2 + \frac{I^2}{2}\left(\frac{\rho}{r}\right)^2 + \frac{c}{k}\log\left(\frac{r}{\rho}\right). \tag{137}$$

The corresponding equation of motion is

$$\ddot{r} + \omega^2(t)r = \frac{I^2}{r^3} - \frac{c\Omega}{k\rho^2 r}. \tag{138}$$

In general, (132) with the restriction (135) is an example of time-dependent potential admitting an invariant quadratic in the velocity [15]. When $c = 0$, equations (136)–(138) are a particular example of the Ermakov system [18–20]. Also, when $c = 0$, J as given by (137) is the Ermakov invariant of the system. Finally, it is possible to reduce the problem to quadratures using the invariants (130) and (137). However, the quadratures cannot be performed analytically.

6. Conclusion

We have found classes of electromagnetic fields for which the planar motion of a non-relativistic test particle is compatible with Lie symmetries. Our procedure was based on the resolution of the basic system of linear first-order partial differential equations (25), (31) and (32) satisfied by the electromagnetic field, using canonical group variables. As shown in section 2, there exist four types of canonical group variables, yielding four classes of electromagnetic fields compatible with Lie symmetry. In comparison with the Noether symmetry analysis [1], an additional dilation invariance term appears in the generator of the symmetries. The dilation invariance is associated with an extra category of electromagnetic field, compatible with Lie symmetries. The electromagnetic fields of sections 4.2 and 4.3 fall into the Noether symmetry framework. The electromagnetic field of section 4.1 can be viewed as a natural extension of the Noether symmetry case treated in section 4.1 of [1]. The class shown in section 4.4 of this paper, however, is essentially new. Its origin can be traced back to the additional dilation invariance, which is not possible in the Noether's theorem framework.

In our treatment, we do not include some symmetries corresponding to the excessively particular classes of electromagnetic fields homogeneous in space. By this, we concentrate on classes of electromagnetic fields depending on arbitrary functions of certain similarity variables involving space coordinates. These classes may be useful, for example, in the search for new exact or approximate solutions for the Vlasov–Maxwell system in collisionless plasma physics. Also, as pointed out in the introduction, symmetry may help in reducing the number of relevant coordinates of the problem and this may represent a considerable reduction in the cost of its numerical treatment.

Finally we analysed the resulting equations of motion to check for the possible existence of first integral and integrability. Unfortunately, and despite the autonomization of the equations of motion in canonical group variables, in general the method does not provide enough constants of motion to guarantee their integrability. In some cases, constants of motion could be constructed by use of the direct method.

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